An improved parametric model for hysteresis loop approximation

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Key features

The model is capable of approximating most of the known types of rate-independent symmetrical hysteresis loops encountered in the practice. The model allows building smooth, piecewise-linear, hybrid, minor, mirror-reflected, inverse, reverse, double and triple hysteresis loops. The error of approximation of a hysteresis loop by the model does not exceed 1% as a rule

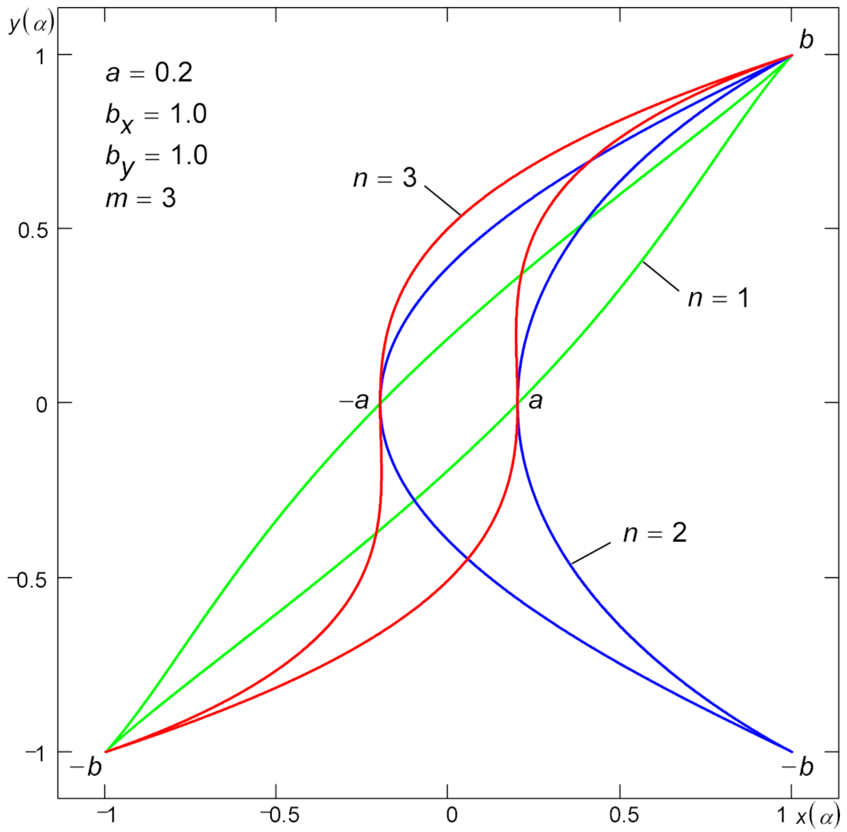
Original model

With the original model, a family of hysteresis loops is described by the following parametric equations

$$x(\alpha) = a\cos^m\alpha + b_x\sin^n\alpha,$$
$$y(\alpha) = b_y\sin\alpha, \qquad (1)$$

where α is a real parameter (α=0…2π); a is x coordinate of the split point; b_x, b_y are the saturation point coordinates; m is a positive odd integer (m=1, 3, 5, …) defining the curvature of the hysteresis loop; n is a positive integer defining the type of the hysteresis loop and its curvature. With n=1, the Leaf loop type is formed; with n=2, 4, 6, … – the Crescent (Boomerang), and with n=3, 5, 7, … – the Classical.

Hysteresis loops supported by the original model



Hysteresis loops of Leaf (n=1), Crescent (Boomerang, n=2), and Classical (n=3) types. The area of all the three loops is the same

Representation as a sum of an unsplit loop and a splitting curve

Hysteresis loop (1) can always be represented as a sum of two parametric curves

$$x(\alpha) = x_1(\alpha) + x_2(\alpha),$$
$$y(\alpha) = y_1(\alpha) + y_2(\alpha), \qquad (2)$$

where $x_1(\alpha)=b_x\sin^n\alpha$, $y_1(\alpha)=b_y\sin\alpha$ is the unsplit loop;
$x_2(\alpha)=a\cos^m\alpha$, $y_2(\alpha)=0$ is the splitting curve.

Representation as a sum of harmonics

The generating function $x(\alpha)$ can easily be expanded in the Fourier series

$$x(\alpha)=\sum_{k=1}^{l}\left(A_k\cos(k\alpha)+B_k\sin(k\alpha)\right),$$
$$y(\alpha)=b_y\sin\alpha, \qquad (3)$$

where the Fourier coefficients A_k, B_k for odd n are determined by the algebraic formulas

$$A_k=\frac{a}{2^{m-1}}C_m^{\frac{m-k}{2}},$$
$$B_k=(-1)^{\left\lfloor\frac{k-1}{2}\right\rfloor}\frac{b_x}{2^{n-1}}C_n^{\frac{n-k}{2}}, \qquad (4)$$

where $C_l^k=l!/[k!(l-k)!]$ is a binomial coefficient.

Representation as a frequency spectrum

Having the Fourier coefficients A_k, B_k, the generating function $x(\alpha)$ can also be represented as

$$x(\alpha) = \sum_{k=1}^{l} Am_k \cos(k\alpha - \varphi_k), \qquad (5)$$

where amplitudes Am_k and phases φ_k of the harmonics are determined by the following formulas

$$Am_k = \sqrt{A_k^2 + B_k^2} = \sqrt{\left(\frac{a}{2^{m-1}} C_m^{\frac{m-k}{2}}\right)^2 + \left(\frac{b_x}{2^{n-1}} C_n^{\frac{n-k}{2}}\right)^2},$$

$$\tan\varphi_k = \frac{B_k}{A_k} = (-1)^{\left\lfloor \frac{k-1}{2} \right\rfloor} 2^{m-n} \frac{C_n^{\frac{n-k}{2}} b_x}{C_m^{\frac{m-k}{2}} a} \qquad (6)$$

Loop tilting by rotation of the coordinate system

The required loop tilting by angle θ at the split point a is performed using the following transformations

$$\begin{aligned} \bar{x}(\alpha) &= x(\alpha) + \sin\theta\left(b_x \sin\theta + b_y \cos\theta\right)\left(\sin\alpha - \sin^n \alpha\right), \\ \bar{y}(\alpha) &= y(\alpha) + \sin\theta\left(b_x \cos\theta - b_y \sin\theta\right)\left(\sin\alpha - \sin^n \alpha\right). \end{aligned} \qquad (7)$$

Area S of the loop (7) is calculated by the formula

$$S = \frac{\pi a}{2^{m-1}} C_m^{\frac{m-1}{2}} \left\{ \sin\theta\left(b_x \cos\theta - b_y \sin\theta\right)\left[1 - (m+1)\prod_{k=0}^{\frac{n-1}{2}} \frac{2k+1}{m+n-2k}\right] + b_y \right\} \qquad (8)$$

Phase shifts $\Delta\alpha$

Phase shifts $\Delta\alpha_1$, $\Delta\alpha_2$, $\Delta\alpha_3$ permit us to tilt the hysteresis loop smoothly by the required angle θ at the split point a as well as to smoothly change the curvature of the loop

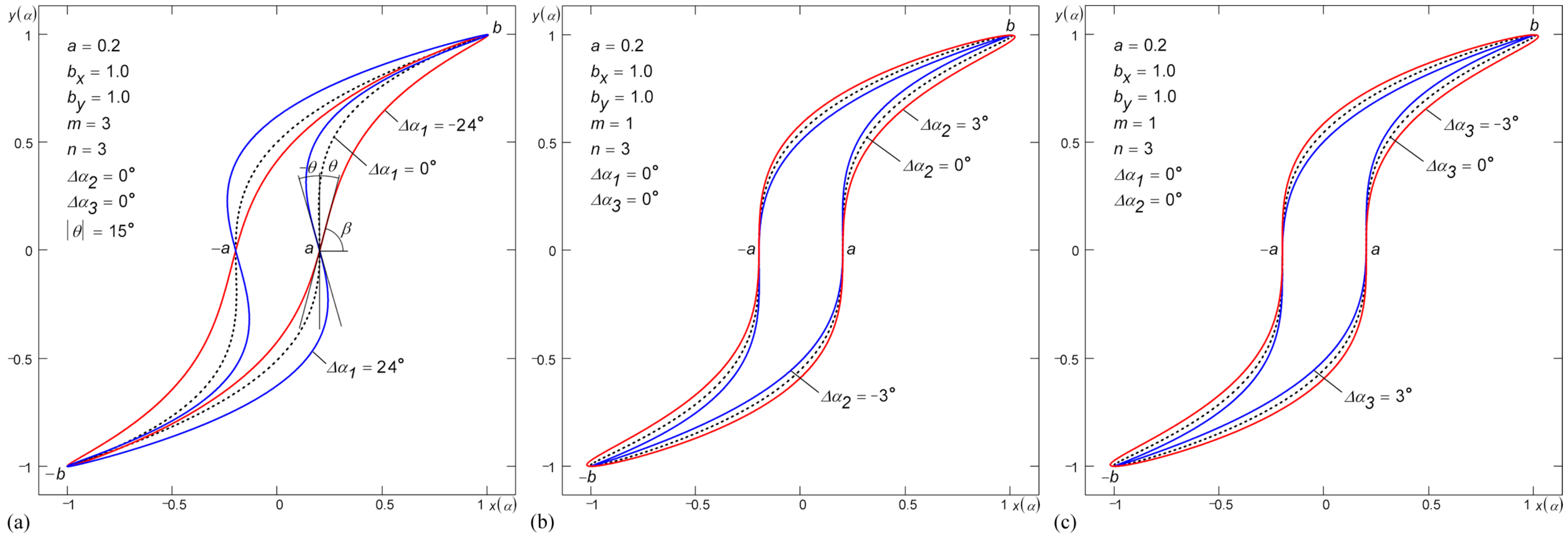
$$x(\alpha) = \hat{a}\cos^m(\alpha + \Delta\alpha_1) + \hat{b}_x \sin^n(\alpha + \Delta\alpha_2),$$
$$y(\alpha) = b_y \sin(\alpha + \Delta\alpha_3), \qquad (9)$$

where $\hat{a}$, $\hat{b}_x$ are corrected parameters of a, b_x, respectively.

The corrected parameters $\hat{a}$, $\hat{b}_x$ can be found by the formulas

$$\hat{a} = \frac{a\cos^n(\Delta\alpha_2 - \Delta\alpha_3) - b_x \sin^n(\Delta\alpha_2 - \Delta\alpha_3)}{\sin^m(\Delta\alpha_1 - \Delta\alpha_3)\sin^n(\Delta\alpha_2 - \Delta\alpha_3) + \cos^m(\Delta\alpha_1 - \Delta\alpha_3)\cos^n(\Delta\alpha_2 - \Delta\alpha_3)},$$
$$\hat{b}_x = \frac{a\sin^m(\Delta\alpha_1 - \Delta\alpha_3) + b_x \cos^m(\Delta\alpha_1 - \Delta\alpha_3)}{\sin^m(\Delta\alpha_1 - \Delta\alpha_3)\sin^n(\Delta\alpha_2 - \Delta\alpha_3) + \cos^m(\Delta\alpha_1 - \Delta\alpha_3)\cos^n(\Delta\alpha_2 - \Delta\alpha_3)} \qquad (10)$$

Effects of the phase shifts $\Delta\alpha$ on the Classical hysteresis loop



(a) Tilting with phase shift $\Delta\alpha_1$, continuous change in the curvature by phase shift (b) $\Delta\alpha_2$, (c) $\Delta\alpha_3$

The Fourier coefficients A_k, B_k when using the phase shifts $\Delta\alpha$

$$A_k = \frac{\hat{a}}{2^{m-1}} C_m^{\frac{m-k}{2}} \cos(k\Delta\alpha_1) + (-1)^{\left\lfloor \frac{k-1}{2} \right\rfloor} \frac{\hat{b}_x}{2^{n-1}} C_n^{\frac{n-k}{2}} \sin(k\Delta\alpha_2),$$

$$B_k = \frac{-\hat{a}}{2^{m-1}} C_m^{\frac{m-k}{2}} \sin(k\Delta\alpha_1) + (-1)^{\left\lfloor \frac{k-1}{2} \right\rfloor} \frac{\hat{b}_x}{2^{n-1}} C_n^{\frac{n-k}{2}} \cos(k\Delta\alpha_2) \qquad (11)$$

Loop area when using the phase shifts $\Delta\alpha$

$$S = \left[\frac{\hat{a}}{2^{m-1}} C_m^{\frac{m-1}{2}} \cos\left(\Delta\alpha_1 - \Delta\alpha_3\right) + \frac{\hat{b}_x}{2^{n-1}} C_n^{\frac{n-1}{2}} \sin\left(\Delta\alpha_2 - \Delta\alpha_3\right) \right] \pi b_y$$

$$= \frac{Am_1\left(\cos\Delta\alpha_3 - \tan\varphi_1 \sin\Delta\alpha_3\right)}{\sqrt{\tan^2\varphi_1 + 1}} \pi b_y \qquad (12)$$

$$= \left(A_1 \cos\Delta\alpha_3 - B_1 \sin\Delta\alpha_3\right)\pi b_y$$

Note that the loop area depends on the amplitude and phase of the 1st harmonic only; the rest of harmonics do not affect the loop area.

Loop tilting and curving by skewing of the coordinate system

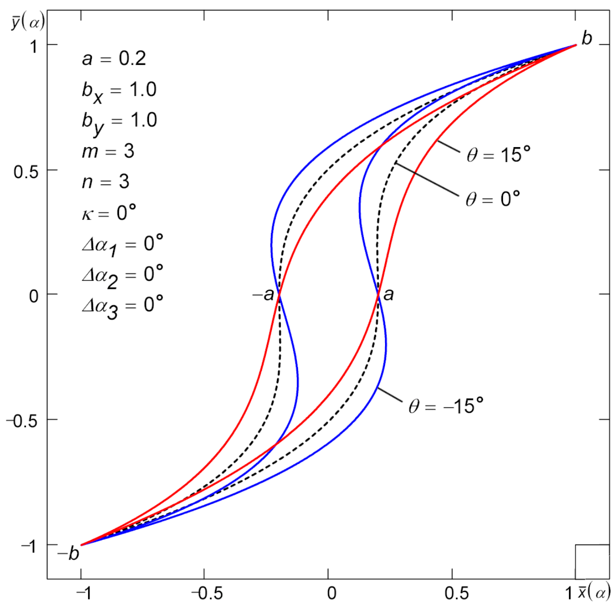
Hysteresis loops can be tilted by skewing of the coordinate system by angle θ along the x axis and by angle κ along the y axis

$$\begin{aligned}\bar{x}(\alpha) &= x(\alpha) + \tan\theta\left(b_x \tan\kappa + b_y\right)\left(\sin\alpha - \sin^n\alpha\right),\\ \bar{y}(\alpha) &= y(\alpha) + b_x \tan\kappa\left(\sin\alpha - \sin^n\alpha\right).\end{aligned} \qquad (13)$$

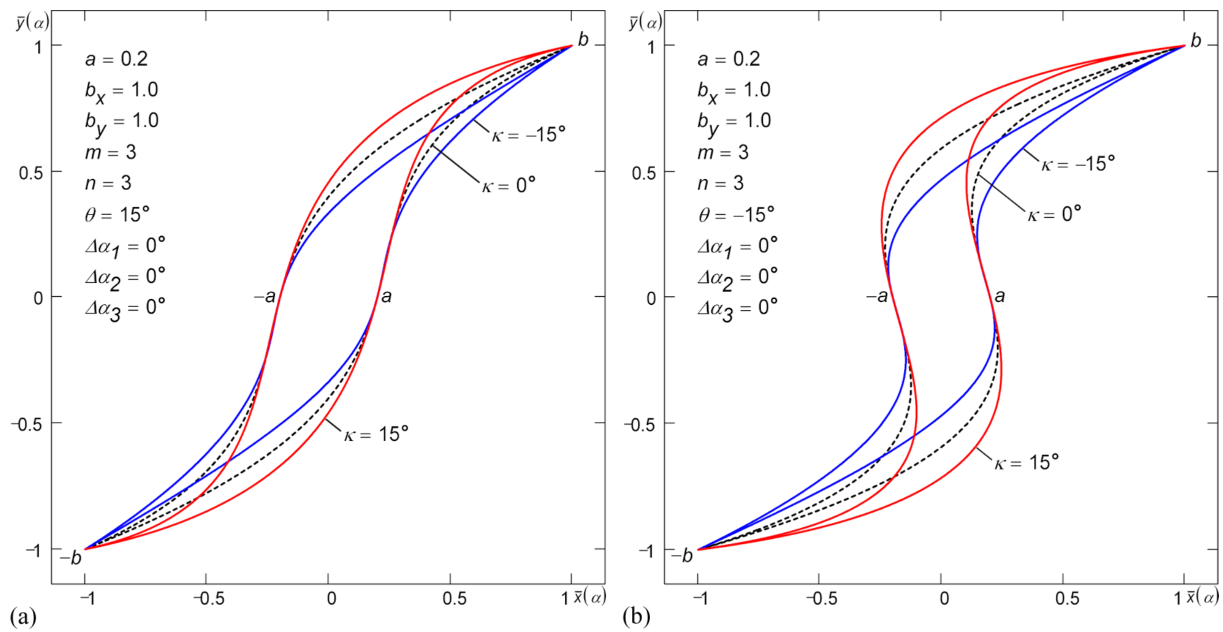
The area of the skewed loop is calculated by the formula

$$S = \frac{\pi a}{2^{m-1}} C_m^{\frac{m-1}{2}} \left\{ b_x \tan\kappa \left[1 - (m+1) \prod_{k=0}^{\frac{n-1}{2}} \frac{2k+1}{m+n-2k} \right] + b_y \right\} \qquad (14)$$

Tilting and curving of the Classical loop by skewing of the coordinate system



Tilting the Classical loop by skewing the coordinate system by angle θ along the x axis



Changing the curvature of the Classical loop by skewing the coordinate system by angle κ along the y axis. Tilting the loop at the split point by angle (a) θ=15°, (b) θ=-15°

Piecewise-linear hysteresis loops built on trapezoidal pulses

Replacing the sine and the cosine in the generating functions $x(\alpha)$, $y(\alpha)$ of model (1) with unit-amplitude trapezoidal pulses $\mathrm{trp_s}$, $\mathrm{trp_c}$, respectively, one can produce piecewise-linear hysteresis loops built on trapezoidal pulses

$$\begin{aligned} x(\alpha) &= a\,\mathrm{trp}_\mathrm{c}^m\,\alpha + (b_x - a)\,\mathrm{trp}_\mathrm{s}^n\,\alpha, \\ y(\alpha) &= b_y\,\mathrm{trp_s}\,\alpha, \end{aligned} \qquad (15)$$

where $\alpha = 0 \ldots T$; T is the pulse period.

Trapezoidal pulses

The trapezoidal pulses trp are defined as follows

$$\mathrm{trp_s}\,\alpha = \sum_i \left[\frac{4}{D-d}\left(\alpha - \frac{T}{2} i\right)(-1)^i \,\mathrm{rect}_1(\alpha, i) + (-1)^i \,\mathrm{rect}_2(\alpha, i)\right],$$
$$\mathrm{trp_c}\,\alpha = \mathrm{trp_s}\left(\alpha + \frac{T}{4}\right), \qquad (16)$$

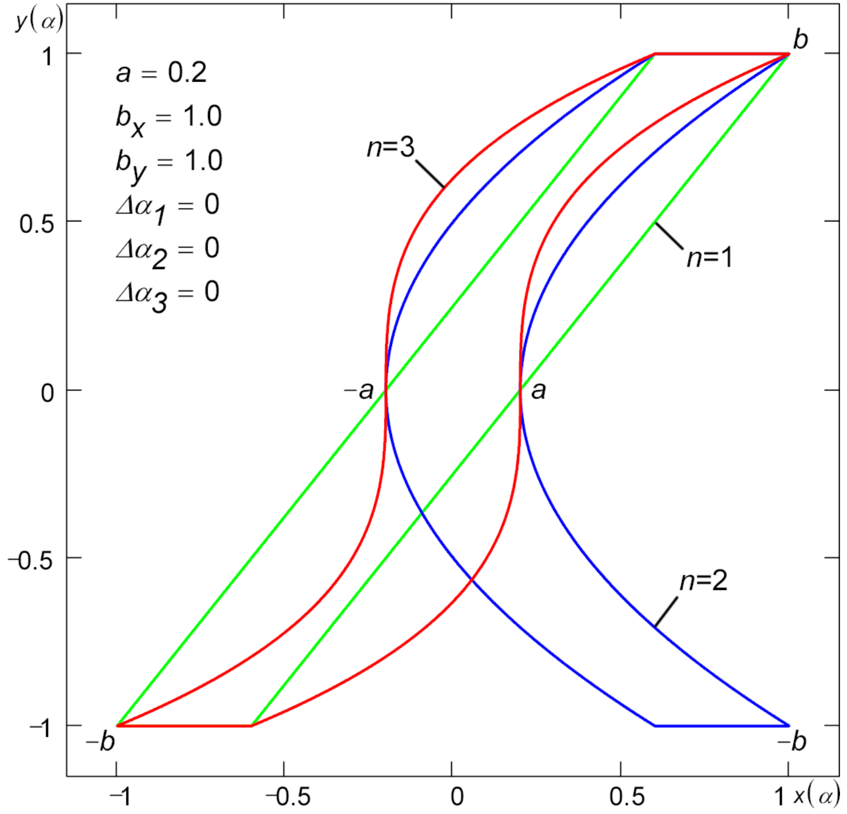
where d, D are the upper and the lower bases of the trapezoidal pulses, respectively ($D=3d$, $T=d+D=4d$); rect_1, rect_2 are rectangular pulses.

Rectangular pulses

The rectangular pulses rect_1, rect_2 are determined by a step function $\text{H}(\alpha)$ (Heaviside function)

$$\text{rect}_1(\alpha,i)=\text{H}\left(\alpha-\frac{T}{2}i+\frac{D-d}{4}\right)-\text{H}\left(\alpha-\frac{T}{2}i-\frac{D-d}{4}\right),$$
$$\text{rect}_2(\alpha,i)=\text{H}\left(\alpha-\frac{T}{2}i-\frac{D-d}{4}\right)-\text{H}\left(\alpha-\frac{T}{2}i-\frac{D+3d}{4}\right) \qquad (17)$$

Piecewise-linear hysteresis loops built on trapezoidal pulses



Piecewise-linear hysteresis loop Leaf (Play without Whiskers, *n*=1), hybrid Crescent (hybrid Boomerang, *n*=2), and hybrid Classical (*n*=3) built on trapezoidal pulses. The area of all the three loops is the same

Piecewise-linear hysteresis loops with phase shifts $\Delta\alpha$

Taking into account the phase shifts $\Delta\alpha_1$, $\Delta\alpha_2$, $\Delta\alpha_3$, equations (15) are written as

$$x(\alpha) = \hat{a}\,\mathrm{trp}_{\mathrm{c}}^{m}(\alpha + \Delta\alpha_1) + \hat{b}_x\,\mathrm{trp}_{\mathrm{s}}^{n}(\alpha + \Delta\alpha_2),$$
$$y(\alpha) = b_y\,\mathrm{trp}_{\mathrm{s}}(\alpha + \Delta\alpha_3). \tag{18}$$

The corrected parameters $\hat{a}$, $\hat{b}_x$ are determined by the following formulas

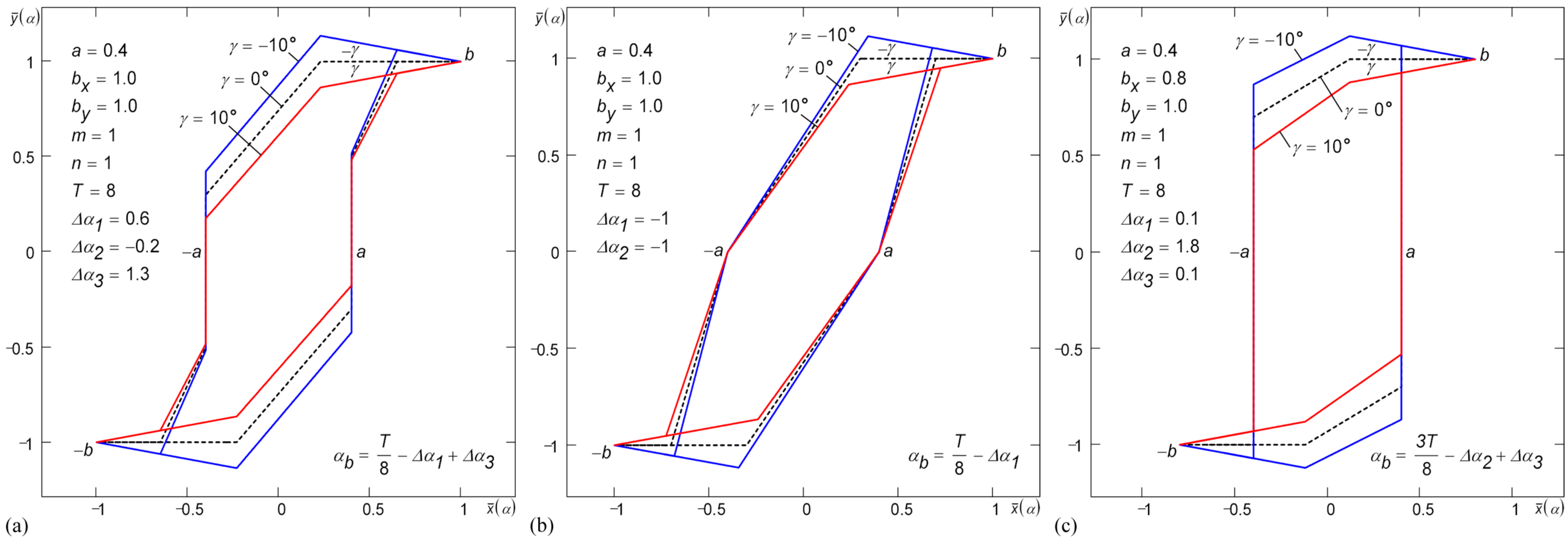
$$\hat{a} = \frac{a\,\mathrm{trp}_{\mathrm{s}}^{n}(\alpha_b + \Delta\alpha_2 - \Delta\alpha_3) - b_x\mathrm{trp}_{\mathrm{s}}^{n}(\Delta\alpha_2 - \Delta\alpha_3)}{\mathrm{trp}_{\mathrm{c}}^{m}(\Delta\alpha_1 - \Delta\alpha_3)\mathrm{trp}_{\mathrm{s}}^{n}(\alpha_b + \Delta\alpha_2 - \Delta\alpha_3) - \mathrm{trp}_{\mathrm{c}}^{m}(\alpha_b + \Delta\alpha_1 - \Delta\alpha_3)\mathrm{trp}_{\mathrm{s}}^{n}(\Delta\alpha_2 - \Delta\alpha_3)},$$
$$\hat{b}_x = \frac{b_x\mathrm{trp}_{\mathrm{c}}^{m}(\Delta\alpha_1 - \Delta\alpha_3) - a\,\mathrm{trp}_{\mathrm{c}}^{m}(\alpha_b + \Delta\alpha_1 - \Delta\alpha_3)}{\mathrm{trp}_{\mathrm{c}}^{m}(\Delta\alpha_1 - \Delta\alpha_3)\mathrm{trp}_{\mathrm{s}}^{n}(\alpha_b + \Delta\alpha_2 - \Delta\alpha_3) - \mathrm{trp}_{\mathrm{c}}^{m}(\alpha_b + \Delta\alpha_1 - \Delta\alpha_3)\mathrm{trp}_{\mathrm{s}}^{n}(\Delta\alpha_2 - \Delta\alpha_3)}. \tag{19}$$

Play-Relay-Play, Play-Play, Play-Relay loops with gain/attenuation

To obtain piecewise-linear loops with gain/attenuation γ, one should add to (18) an extra term (curve) responsible for gain/attenuation (m=n=1)

$$\begin{aligned} \bar{x}(\alpha) &= x(\alpha), \\ \bar{y}(\alpha) &= y(\alpha) + \tan\gamma\left[x(\alpha) - b_x \operatorname{trp}_s(\alpha + \Delta\alpha_3)\right] \end{aligned} \qquad (20)$$

Piecewise-linear hysteresis loops Play-Relay-Play, Play-Play, Play-Relay



Piecewise-linear hysteresis loops with gain/attenuation γ. (a) Play-Relay-Play, (b) Play-Play, (c) Play-Relay built on trapezoidal pulses using the phase shifts

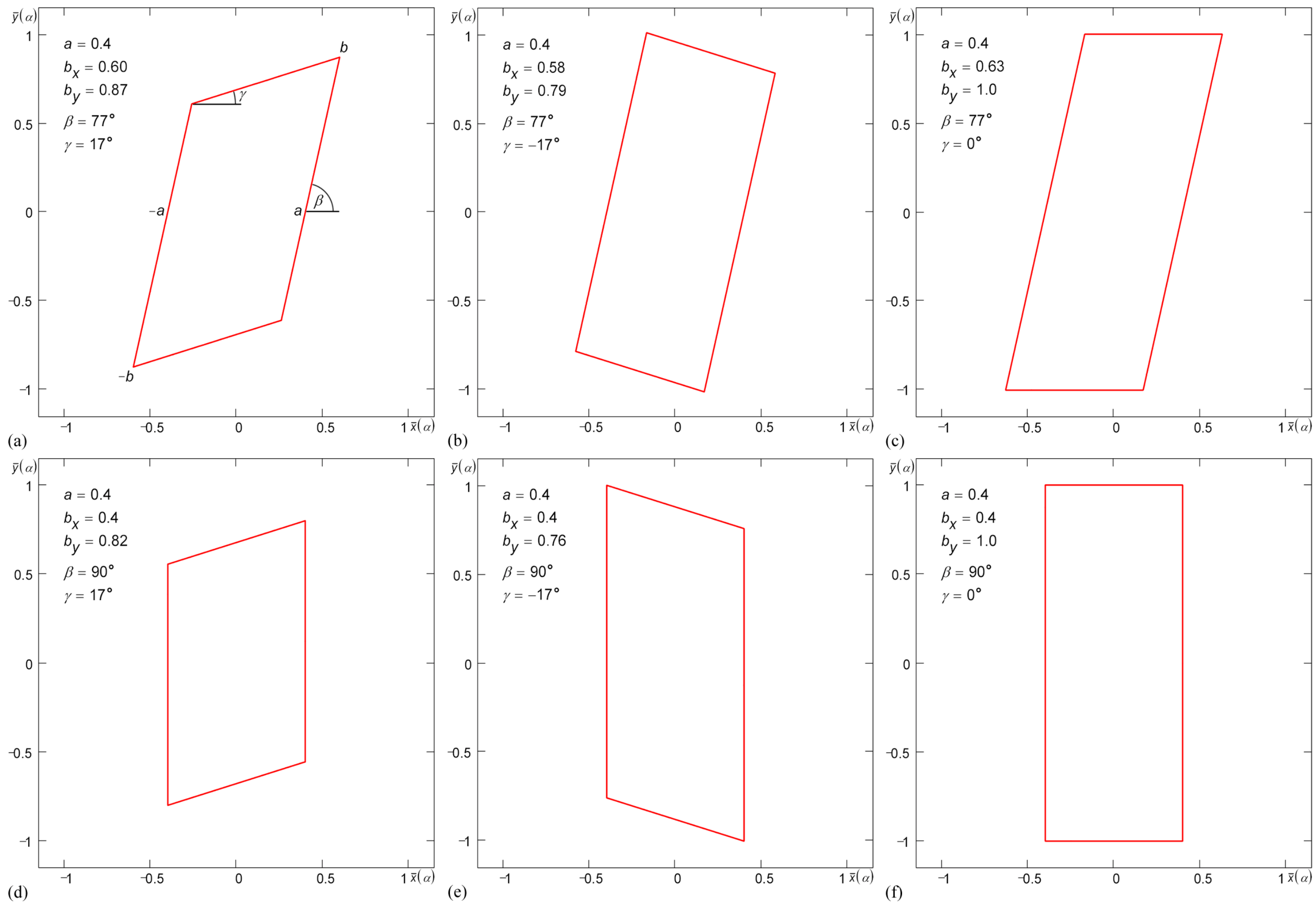
Whiskerless Play and Non-ideal Relay loops with gain/attenuation

Whiskerless Play and Non-ideal Relay loops with gain/attenuation are built by the following equations

$$\bar{x}(\alpha) = \frac{a \tan\beta \operatorname{trp_c} \alpha + \left(b_y - b_x \tan\gamma\right) \operatorname{trp_s} \alpha}{\tan\beta - \tan\gamma},$$
$$\bar{y}(\alpha) = \tan\beta\left(\bar{x}(\alpha) - a \operatorname{trp_c} \alpha\right), \qquad (21)$$

where the loop parameters a, b_x, b_y, β are interrelated as $\tan\beta = b_y/(b_x - a)$.

Whiskerless Play and Non-ideal Relay loops built on trapezoidal pulses



Area of Play and Non-ideal Relay loops

The areas of the simplest piecewise-linear loops (21) having a parallelogram shape are determined by the formula

$$S = 4a\left[\frac{(a-b_x)\tan\gamma + b_y}{\tan\beta - \tan\gamma}\tan\beta - \frac{a\sin\beta\sin\gamma}{\sin(\beta-\gamma)}\right] \qquad (22)$$

Whiskerless hybrid Classical hysteresis loop

Whiskerless hybrid Classical loops with the desired slope $\beta = \pi/2 - \theta$ at the split point are built according to the following transformation

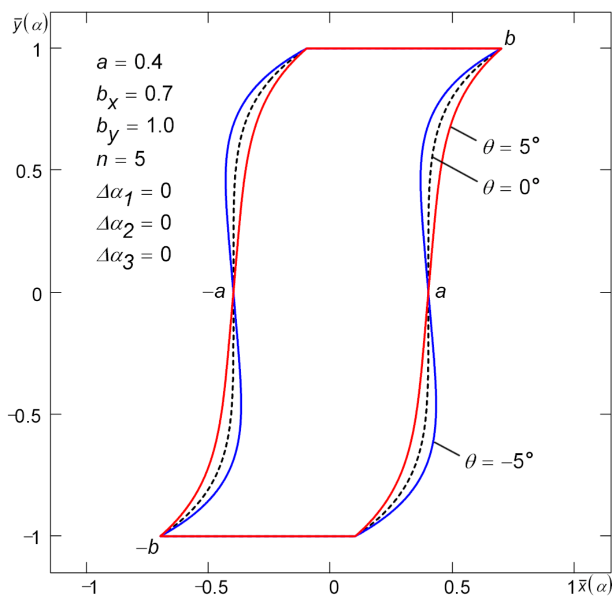
$$\begin{aligned}\bar{x}(\alpha) &= x(\alpha) + b_y \sin\theta\left(\mathrm{trp_s}\,\alpha - \mathrm{trp_s^{\mathit{n}}}\,\alpha\right),\\ \bar{y}(\alpha) &= y(\alpha).\end{aligned} \qquad (23)$$

Whiskerless hybrid Classical loops having the required slope β at the split point, the required inclination (gain/attenuation) γ of the rectilinear section, and the required curvature κ of the curvilinear section are built according to the following transformation

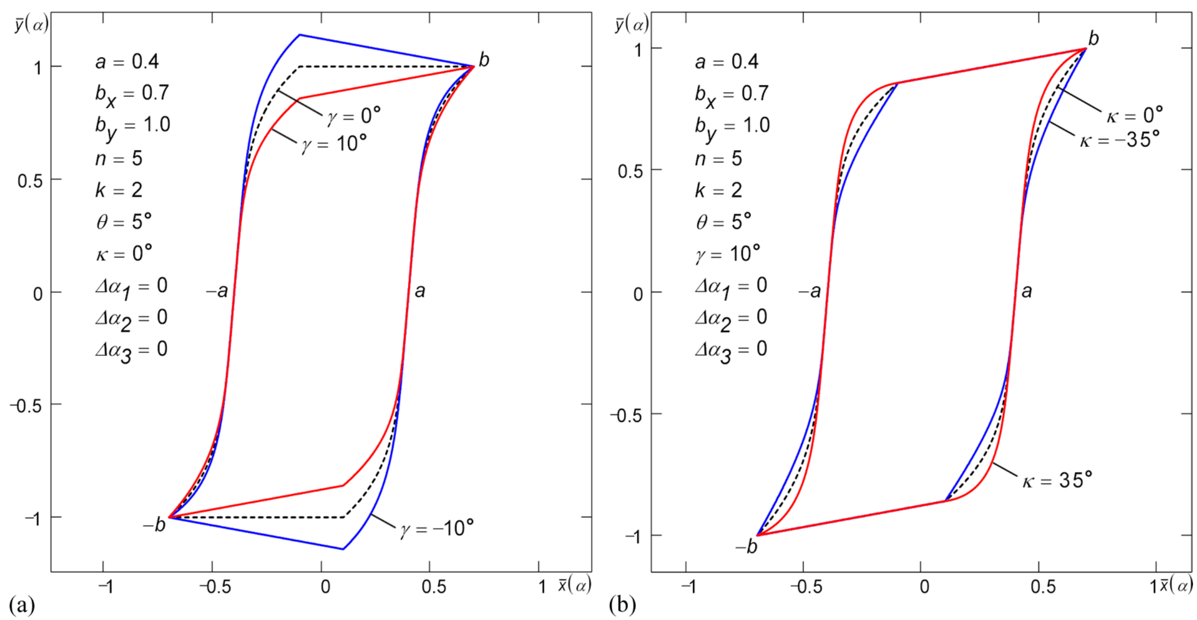
$$\begin{aligned}\bar{x}(\alpha) &= x(\alpha) + \tan\theta\left[(b_x - a)\tan\kappa - b_x\tan\gamma + b_y\right]\left(\mathrm{trp_s}\,\alpha - \mathrm{trp_s^{\mathit{n}}}\,\alpha\right),\\ \bar{y}(\alpha) &= y(\alpha) + \left[(b_x - a)\tan\kappa - b_x\tan\gamma\right]\left(\mathrm{trp_s}\,\alpha - \mathrm{trp_s^{\mathit{n}}}\,\alpha\right) + a\tan\gamma\left(\mathrm{trp_c}\,\alpha\,\mathrm{trp_s^{\mathit{k}}}\,\alpha - \mathrm{trp_s^{\mathit{n}}}\,\alpha\right),\end{aligned} \qquad (24)$$

where k=2, 4, ... is an additional parameter to control the loop curvature.

Hybrid Classical whiskerless hysteresis loops built on trapezoidal pulses



Hybrid Classical loops with the specified slope $\beta=\pi/2-\theta$ at the split point



Hybrid Classical loops with specified slope $\beta=\pi/2-\theta$, gain/attenuation γ, and curvature κ. (a) Various gains γ for fixed β and κ, (b) various curvatures κ for fixed β and γ

Area of hybrid Classical loop

The area of hybrid Classical loop (23) is calculated by the formula

$$S = 4ab_y. \qquad (25)$$

The area of hybrid Classical loop with non-zero gain/attenuation (24) is calculated by the formula

$$S = 4a\left\{\begin{aligned}&\left[(b_x - a)\tan\kappa - b_x\tan\gamma + b_y\right]\left[\frac{k(n-1)}{(k+1)(n+k)}\tan\theta\tan\gamma + 1\right]\\&+(b_x - a)\left(\frac{k}{n+k}\tan\gamma - \tan\kappa\right)\end{aligned}\right\} \qquad (26)$$

Loops built on triangular pulses

Besides trapezoidal pulses trp, formulas (9) can operate with triangular pulses tri, which are particular cases of the trapezoidal pulses (d=0, T=D)

$$x(\alpha) = \hat{a}\,\mathrm{tri}_c^m(\alpha + \Delta\alpha_1) + \hat{b}_x\,\mathrm{tri}_s^n(\alpha + \Delta\alpha_2),$$
$$y(\alpha) = b_y\,\mathrm{tri}_s(\alpha + \Delta\alpha_3). \quad (27)$$

The corrected parameters $\hat{a}$, $\hat{b}_x$ are determined by the following formulas

$$\hat{a} = \frac{a\,\mathrm{tri}_c^n(\Delta\alpha_2 - \Delta\alpha_3) - b_x\mathrm{tri}_s^n(\Delta\alpha_2 - \Delta\alpha_3)}{\mathrm{tri}_s^m(\Delta\alpha_1 - \Delta\alpha_3)\mathrm{tri}_s^n(\Delta\alpha_2 - \Delta\alpha_3) + \mathrm{tri}_c^m(\Delta\alpha_1 - \Delta\alpha_3)\mathrm{tri}_c^n(\Delta\alpha_2 - \Delta\alpha_3)},$$
$$\hat{b}_x = \frac{a\,\mathrm{tri}_s^m(\Delta\alpha_1 - \Delta\alpha_3) + b_x\mathrm{tri}_c^m(\Delta\alpha_1 - \Delta\alpha_3)}{\mathrm{tri}_s^m(\Delta\alpha_1 - \Delta\alpha_3)\mathrm{tri}_s^n(\Delta\alpha_2 - \Delta\alpha_3) + \mathrm{tri}_c^m(\Delta\alpha_1 - \Delta\alpha_3)\mathrm{tri}_c^n(\Delta\alpha_2 - \Delta\alpha_3)} \quad (28)$$

Triangular pulses

The triangular pulses tri are defined as follows

$$\mathrm{tri}_s\,\alpha = \frac{4}{T}\sum_i\left(\alpha - \frac{T}{2}i\right)(-1)^i\,\mathrm{rect}(\alpha, i),$$

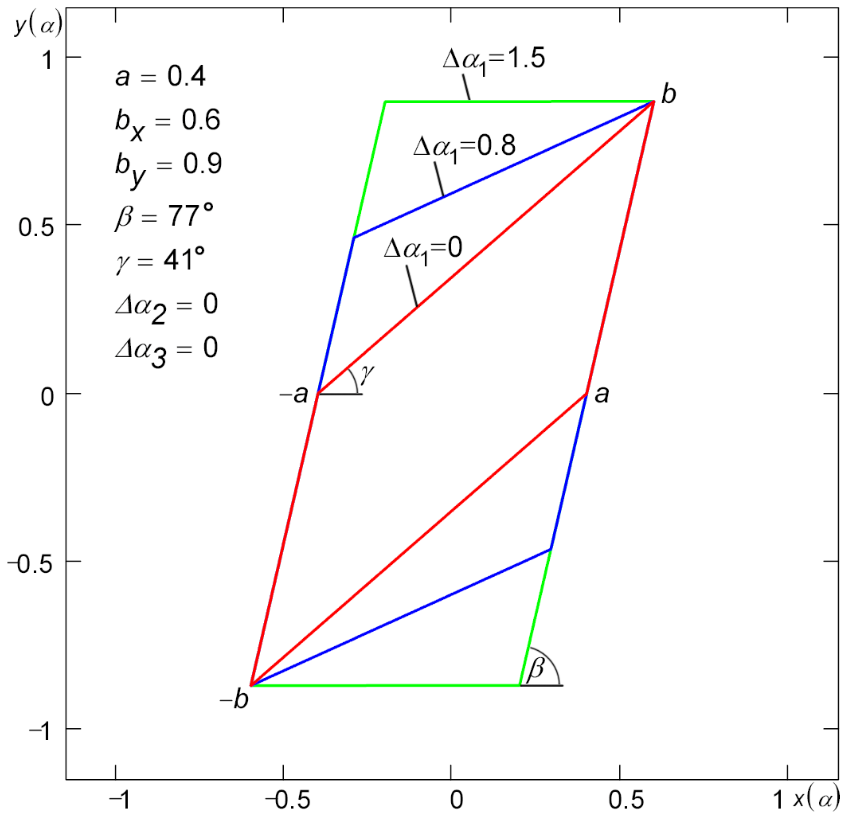
$$\mathrm{tri}_c\,\alpha = \mathrm{tri}_s\left(\alpha + \frac{T}{4}\right), \qquad (29)$$

where rect are rectangular pulses.

The rectangular pulses rect are determined by a step function H(α)

$$\mathrm{rect}(\alpha, i) = \mathrm{H}\left(\alpha - \frac{T}{2}i + \frac{T}{4}\right) - \mathrm{H}\left(\alpha - \frac{T}{2}i - \frac{T}{4}\right) \qquad (30)$$

Piecewise-linear hysteresis loop Leaf built on triangular pulses



Piecewise-linear loop Leaf (bi-linear loop, Play without Whiskers) at different values of phase shift $\Delta\alpha_1$ built on triangular pulses

General expression for a piecewise-linear loops Play and Non-ideal Relay

The general expression describing a piecewise-linear hysteresis loop Play with Gain is as follows

$$x(\alpha) = b_x \mathrm{tri}_s\, \alpha,$$

$$y(\alpha) = (b_y - b_x \tan\gamma)\mathrm{trp}_s\left(\alpha - \frac{\alpha_a \tan\beta}{\tan\beta - \tan\gamma}\right) + b_x \tan\gamma\, \mathrm{tri}_s\, \alpha, \tag{31}$$

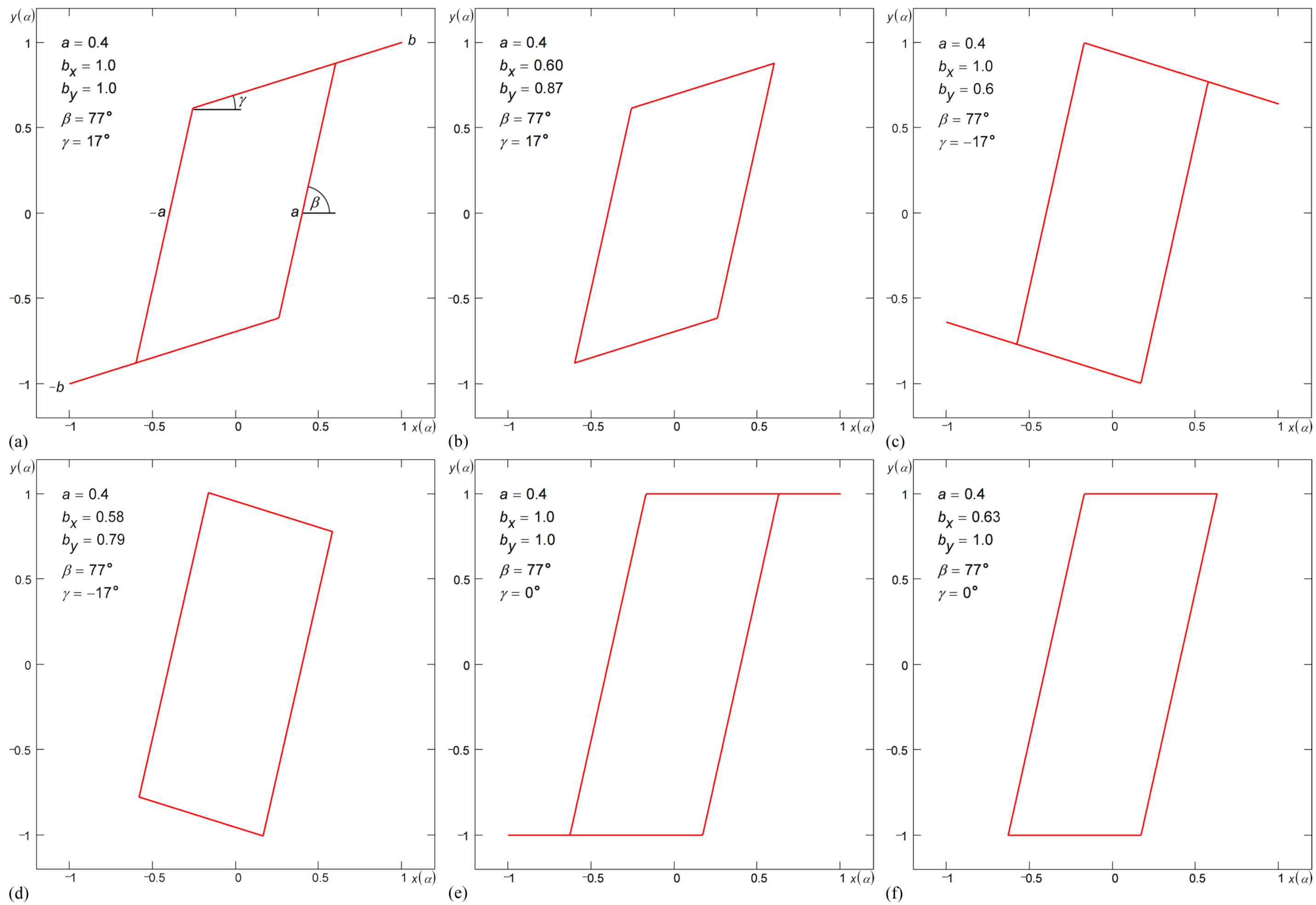
where $\alpha_a = aT/(4b_x)$ is the value of parameter α at split point a.

The upper base d of the trapezoidal pulses trp_s in (31) is determined according to the formula

$$d = \frac{T(b_x \tan\beta - b_y)}{2b_x(\tan\beta - \tan\gamma)}, \tag{32}$$

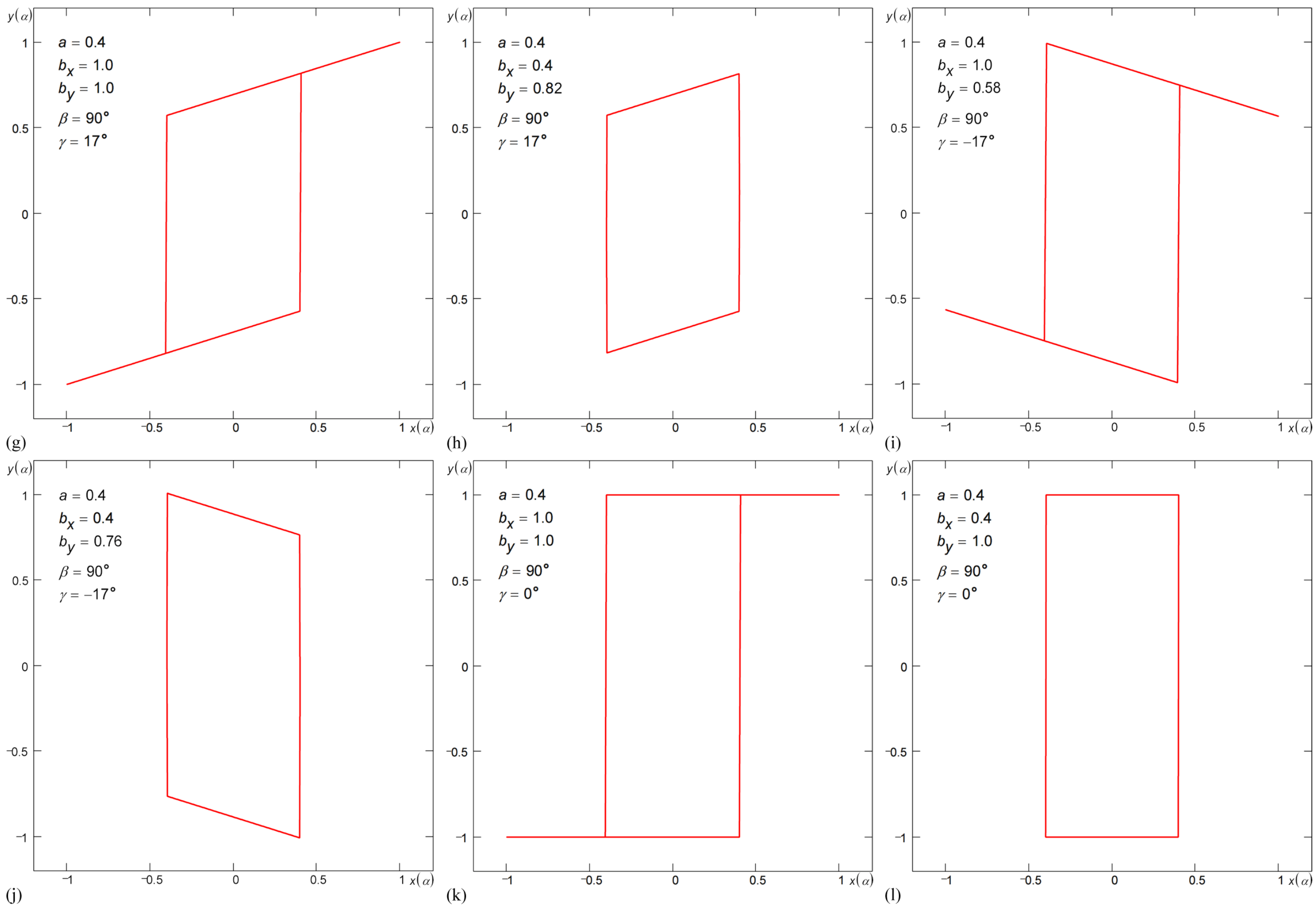
and the lower base D according to the formula $D = T - d$.

Piecewise-linear Play hysteresis loops built on triangular pulses



(a) Play with Gain, (b) Play with Gain w/o Whiskers (parallelogram loop), (c) Play with Attenuation, (d) Play with Attenuation w/o Whiskers, (e) Play (backlash), (f) Play w/o Whiskers

Piecewise-linear Relay hysteresis loops built on triangular pulses



(g) Non-ideal Relay with Gain, (h) Non-ideal Relay with Gain w/o Whiskers, (i) Non-ideal Relay with Attenuation, (j) Non-ideal Relay with Attenuation w/o Whiskers, (k) Non-ideal Relay (Schmitt trigger), (l) Non-ideal Relay w/o Whiskers (rectangular loop)

Double non-self-crossing hysteresis loop

The equations for a double smooth loop non-self-crossing in the origin of coordinates (0-shaped loop) are as follows (α=0…2π)

$$\bar{\bar{x}}(\alpha) = x\left(2\alpha - \frac{\pi}{2}\operatorname{sgn}(\pi-\alpha) - \Delta\alpha_3\right) + b_x \operatorname{sgn}(\pi-\alpha)$$

$$= (2\operatorname{rect}\alpha - 1)\left(x\left(2\alpha - \Delta\alpha_3 - \frac{\pi}{2}\right) + b_x\right),$$

$$\bar{\bar{y}}(\alpha) = y\left(2\alpha - \frac{\pi}{2}\operatorname{sgn}(\pi-\alpha) - \Delta\alpha_3\right) + b_y \operatorname{sgn}(\pi-\alpha)$$

$$= (2\operatorname{rect}\alpha - 1)\left(y\left(2\alpha - \Delta\alpha_3 - \frac{\pi}{2}\right) + b_y\right),$$

(33)

where sgnα=α/|α| is the signum function; rectα=H(α)-H(α-π) is a π-wide rectangular pulse.

Double self-crossing hysteresis loop

The loop with a self-crossing in the origin of coordinates (8-shaped loop) can be built according to the formulas (α=0…2π)

$$\bar{\bar{x}}(\alpha) = x\left(\left(2\alpha - \frac{\pi}{2}\right)\operatorname{sgn}(\pi - \alpha) - \Delta\alpha_3\right) + b_x \operatorname{sgn}(\pi - \alpha)$$

$$= \operatorname{rect}\alpha\left(x\left(2\alpha - \Delta\alpha_3 - \frac{\pi}{2}\right) + b_x\right) + (1 - \operatorname{rect}\alpha)\left(x\left(\frac{\pi}{2} - \Delta\alpha_3 - 2\alpha\right) - b_x\right),$$

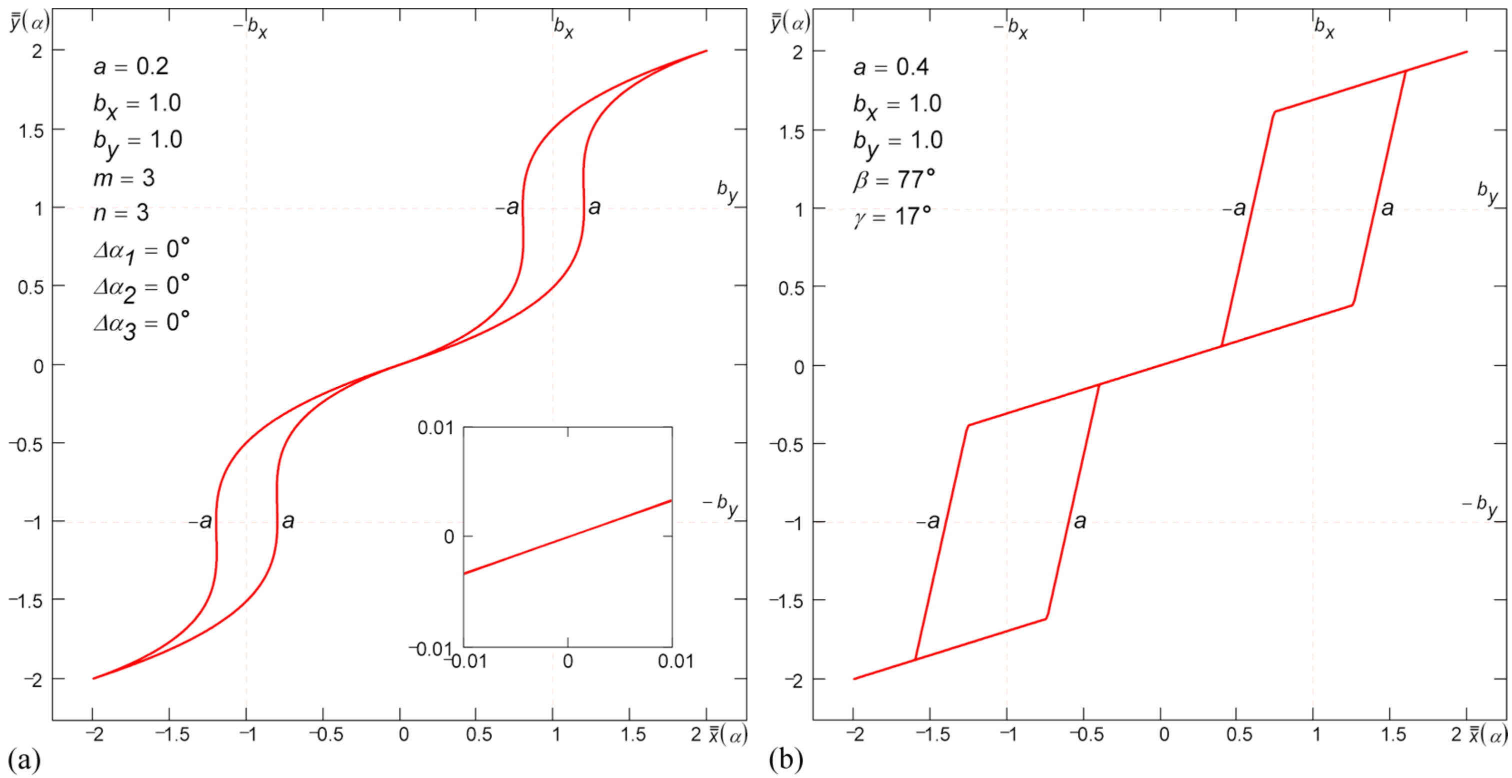
(34)

$$\bar{\bar{y}}(\alpha) = y\left(\left(2\alpha - \frac{\pi}{2}\right)\operatorname{sgn}(\pi - \alpha) - \Delta\alpha_3\right) + b_y \operatorname{sgn}(\pi - \alpha)$$

$$= \operatorname{rect}\alpha\left(y\left(2\alpha - \Delta\alpha_3 - \frac{\pi}{2}\right) + b_y\right) + (1 - \operatorname{rect}\alpha)\left(y\left(\frac{\pi}{2} - \Delta\alpha_3 - 2\alpha\right) - b_y\right).$$

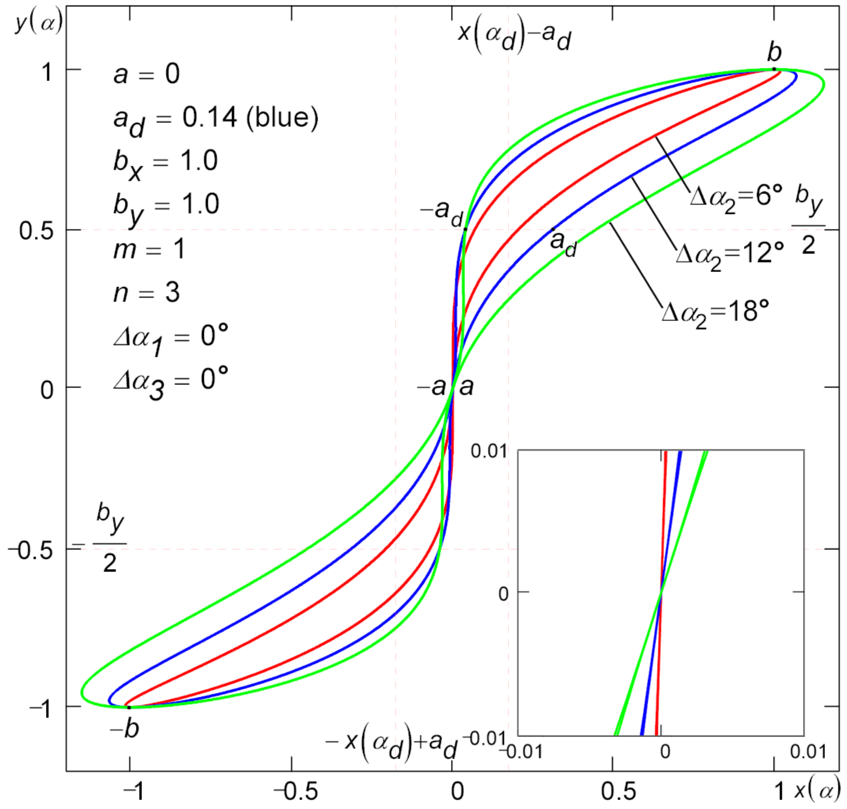
The loops (33), (34) have no appearance differences. By replacing π with *T*/2 (α=0…*T*), double piecewise-linear loops are built by formulas (33), (34).

Examples of double hysteresis loops



Double (a) smooth, (b) piecewise-linear hysteresis loop formed by linking two loops (a) Classical, (b) Play with Gain in the saturation point *b*

Pinching a loop in the origin of coordinates



Double non-self-crossing hysteresis loop of Propeller type formed by pinching a loop at the point of coordinate origin by means of zero splitting *a* and a phase shift $\Delta\alpha_2$

Triple hysteresis loops

A triple loop is assembled of three loops – one central loop and two outside loops linked at the saturation points *b* (α=0…2π)

$$\begin{aligned}\bar{\bar{\bar{x}}}(\alpha) &= (\operatorname{rect}\alpha + \operatorname{rect}(\alpha-\pi))x_1\left(3\alpha - \frac{\pi}{2}\right)\\ &+ (\operatorname{rect}\alpha + \operatorname{rect}(\alpha-\pi)-1)\left[x_2\left(\pm 3\alpha - \frac{\pi}{2}\right) - (b_{1x}+b_{2x})\operatorname{sgn}(\pi-\alpha)\right],\\ \bar{\bar{\bar{y}}}(\alpha) &= (\operatorname{rect}\alpha + \operatorname{rect}(\alpha-\pi))y_1\left(3\alpha - \frac{\pi}{2}\right)\\ &+ (\operatorname{rect}\alpha + \operatorname{rect}(\alpha-\pi)-1)\left[y_2\left(\pm 3\alpha - \frac{\pi}{2}\right) - (b_{1y}+b_{2y})\operatorname{sgn}(\pi-\alpha)\right],\end{aligned} \tag{35}$$

where $x_1(\alpha)$, $y_1(\alpha)$ are equations of the central loop; $x_2(\alpha)$, $y_2(\alpha)$ are equations of the outside loops; b_{1x}, b_{1y} are coordinates of the saturation point of the central loop; b_{2x}, b_{2y} are coordinates of the saturation points of the outside loops; rectα=H(α)-H(α-π/3) is a π/3-wide rectangular pulse.

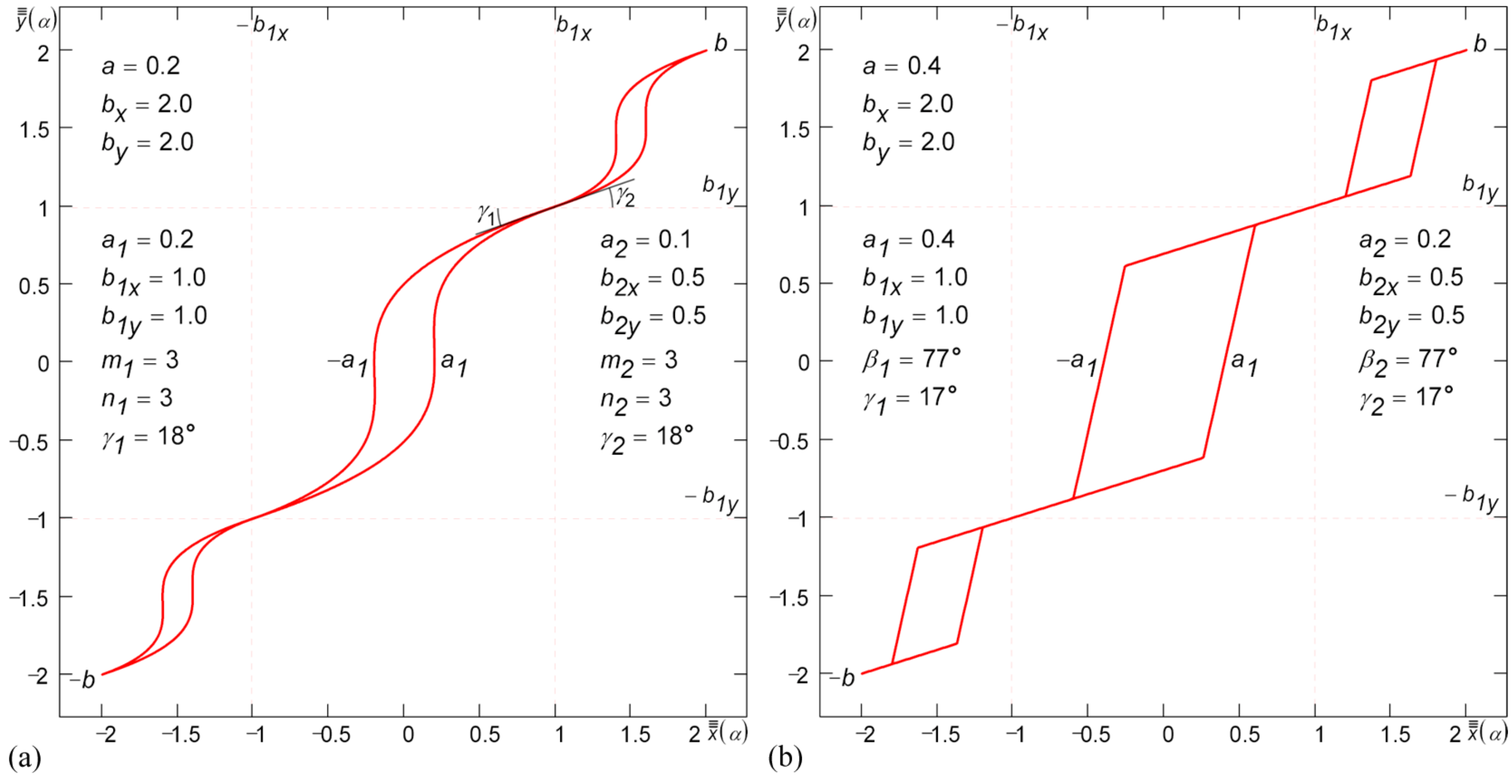
Condition for triple loop assembling

When assembling loops (35), the condition $\gamma_1 = \gamma_2$ is usually met, where γ_1, γ_2 are slope angles of tangents to the unsplit (a_1=0) central loop and the unsplit (a_2=0) outside loops, respectively, at the saturation point b_1. The slope angles γ_1, γ_2 of the tangents are defined by the formulas

$$\gamma_1 = \arctan\frac{b_{1y}}{n_1 b_{1x}},$$
$$\gamma_2 = \arctan\frac{b_{2y}}{n_2 b_{2x}}. \qquad (36)$$

In case the argument 3α of functions $x_2(\alpha)$, $y_2(\alpha)$ in equations (35) is used with the plus sign, a non-self-crossing loop is obtained, and in the case of the minus sign – a self-crossing loop; both the loops have the same appearance. Triple non-self-crossing and self-crossing piecewise-linear loops are built by formulas (35) by replacing π with $T/2$ (α=0…T).

Examples of triple hysteresis loops



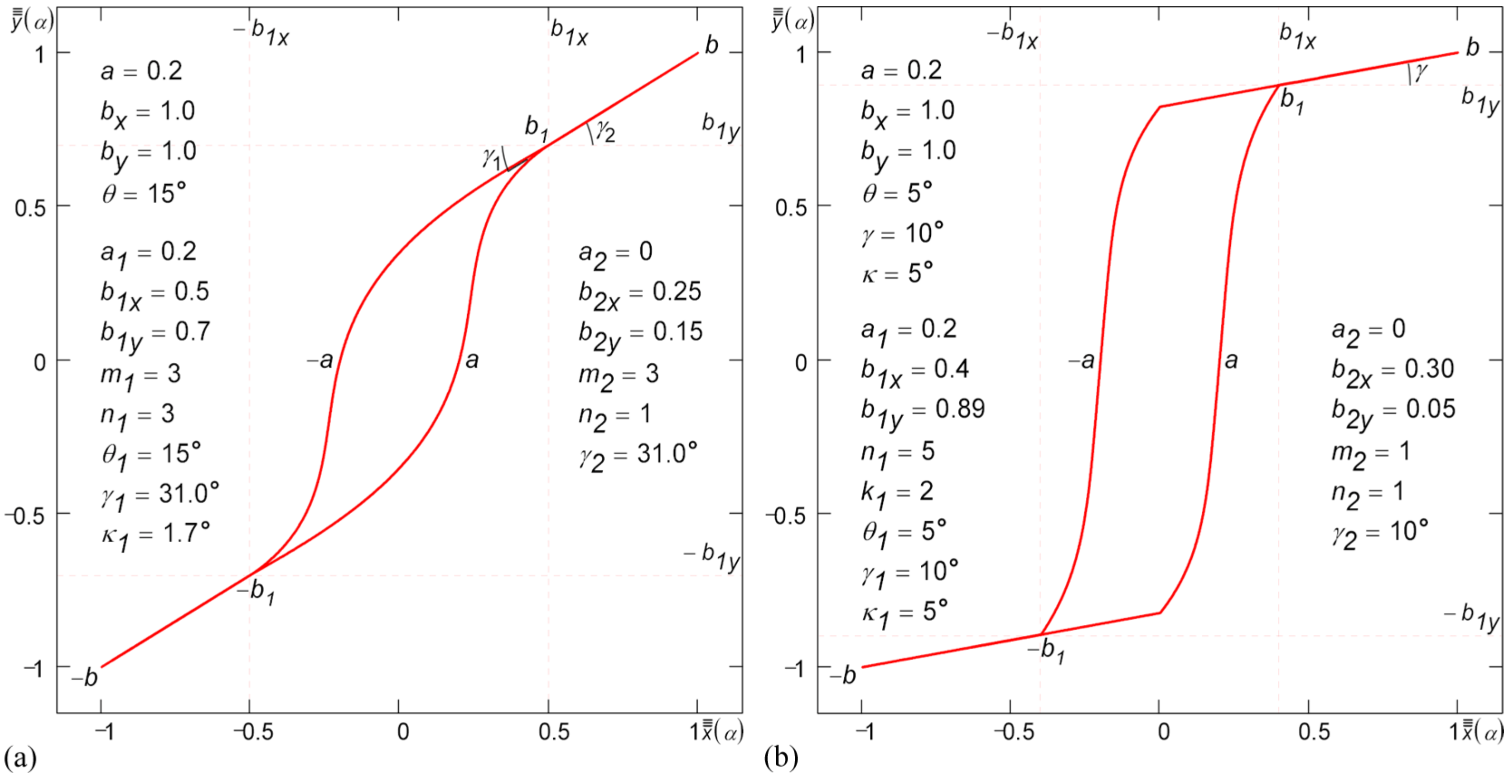
Triple (a) smooth, (b) piecewise-linear hysteresis loop formed by linking three loops (a) Classical, (b) Play with Gain in the saturation points *b*. In the linking points, the loop can be made both non-self-crossing and self-crossing

Loops with arbitrarily long whiskers

Triple hysteresis loops (35) are useful for producing single smooth or single hybrid hysteresis loops having long whiskers. The triple loop consists of the central Tilted Classical loop and whiskers, which are formed from a pair of the outside unsplit loops of Leaf type oriented at the angle γ_2. To obtain hybrid Classical loop with whiskers, the following equations can be used (α=0…T)

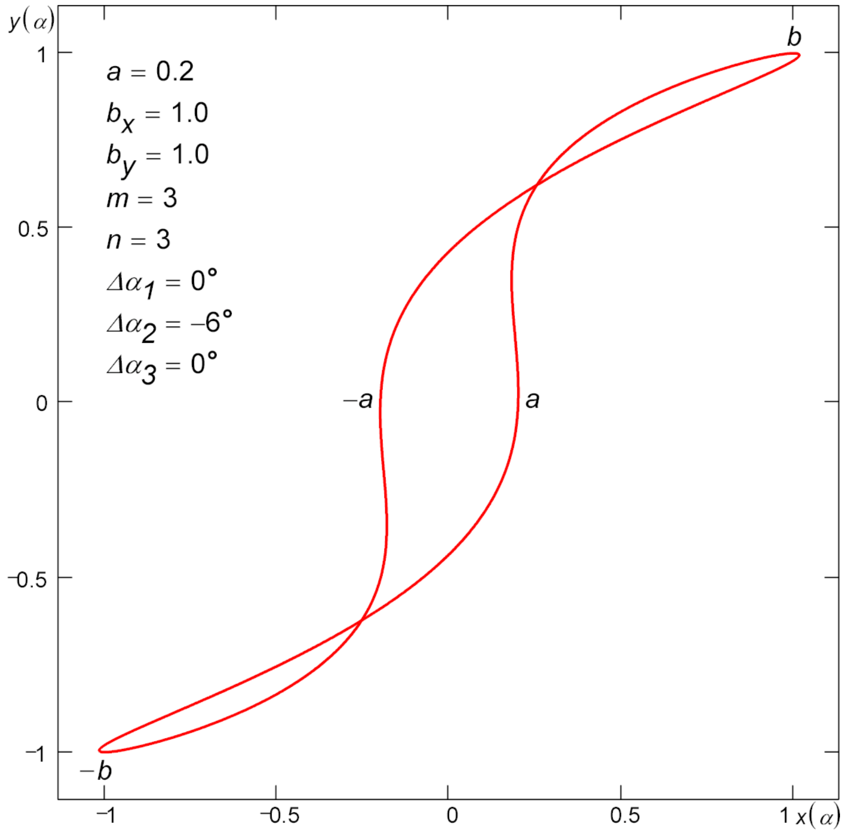
$$\begin{aligned}
\overline{\overline{\overline{x}}}(\alpha) &= \left(\operatorname{rect}\alpha + \operatorname{rect}\left(\alpha - \frac{T}{2}\right)\right) x_1\left(3\alpha - \frac{3T}{8}\right) \\
&+ \left(\operatorname{rect}\alpha + \operatorname{rect}\left(\alpha - \frac{T}{2}\right) - 1\right)\left[x_2\left(3\alpha - \frac{T}{4}\right) - (b_{1x} + b_{2x})\operatorname{sgn}\left(\frac{T}{2} - \alpha\right)\right], \\
\overline{\overline{\overline{y}}}(\alpha) &= \left(\operatorname{rect}\alpha + \operatorname{rect}\left(\alpha - \frac{T}{2}\right)\right) y_1\left(3\alpha - \frac{3T}{8}\right) \\
&+ \left(\operatorname{rect}\alpha + \operatorname{rect}\left(\alpha - \frac{T}{2}\right) - 1\right)\left[y_2\left(3\alpha - \frac{T}{4}\right) - (b_{1y} + b_{2y})\operatorname{sgn}\left(\frac{T}{2} - \alpha\right)\right]
\end{aligned} \qquad (37)$$

Examples of loops having arbitrarily long whiskers



Simulation of a single Classical (a) smooth, (b) hybrid loop with long whiskers by means of a triple non-self-crossing hysteresis loop. Whiskers are the outside pair of (a) smooth, (b) piecewise-linear unsplit loops of Leaf type

Squeeze causing a foldover



The triple self-crossing loops are formed by setting up a negative phase shift $\Delta\alpha_2$ (or positive $\Delta\alpha_3$) that "squeezes" the loop so tight that a foldover appears

Further reading

- R. V. Lapshin, “An improved parametric model for hysteresis loop approximation”, Review of Scientific Instruments, vol. 91, iss. 6, no. 065106, 31 pp., 2020 (DOI: 10.1063/5.0012931)
- R. V. Lapshin, “Analytical model for the approximation of hysteresis loop and its application to the scanning tunneling microscope”, Review of Scientific Instruments, vol. 66, iss. 9, pp. 4718-4730, 1995 (DOI: 10.1063/1.1145314)

Supplementary material

- R. V. Lapshin, “Hysteresis loop”, Mathcad 2001i worksheets, ver. 03.01.2020
- R. V. Lapshin, “Hysteresis loop”, Readable Mathcad 2001i worksheets, ver. 03.01.2020